# C3-R4: MATHEMATICAL METHODS FOR COMPUTING

**NOTE:**

| | |
|---|---|
| **1.** | **Answer question 1 and any FOUR from questions 2 to 7.** |
| **2.** | **Parts of the same question should be answered together and in the same sequence.** |

**Time: 3 Hours** **Total Marks: 100**

**1.**

a) A computer assembling company receives 24% of parts from supplier A, 36% of parts from supplier B, and the remaining 40% parts from supplier C. Five percent of parts supplied by A, ten percent of parts supplied by B, and six percent of parts supplied by C are defective. If an assembled computer has a defective part in it, What is the probability that this part was received from Supplier A?

b) Two discrete random variables *X* and *Y* have joint probability mass function given by the following table:

| | | Y: 1 | 2 | 3 |
|---|---|---|---|---|
| X | 1 | 1/12 | 1/6 | 1/12 |
| | 2 | 1/6 | 1/4 | 1/12 |
| | 3 | 1/12 | 1/12 | 0 |

Compute the probability of the event: *X* is odd

c) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and with transition matrix

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

and initial distribution $P(X_0=i)=\frac{1}{3}$, i=0, 1, 2.

Find $P(X_2=2, X_1=1 \mid X_0=2)$.

d) A telephone exchange receives 100 calls per minute on average, according to a Poisson process. What is the probability that no calls are received in an interval of 5 seconds?

e) Write the dual of the following linear programming problem:

Maximize $Z = 2x_1 + 3x_2$

Subject to

$$x_1 + 2x_2 \leq 4$$
$$2x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

f) Laplace transforms of $f_1(t)$ and $f_2(t)$ are given by $\bar{f}_1(s)$ and $\bar{f}_2(s)$. Find the Laplace transform of $a_1 f_1(t) + a_2 f_2(t)$
where $a_1$ and $a_2$ are constants.

g) Let *X* be a random variable that takes on 5 possible values with respective probabilities 0.35, 0.2, 0.2, 0.2, 0.05. Find the entropy H(X) of random variable X.

**(7×4)**

**2.**

a) Arrival rate of telephone calls at a telephone booth is according to Poisson process with an average time of 9 minutes between consecutive arrivals. The length of telephone call is exponentially distributed with a mean of 3 minutes. Compute the probability that a person arriving at the booth will have to wait for more than 10 minutes before the phone is free.

b) Formulate a set of differential equations describing M/M/1 queuing system. Assuming queue is in equilibrium, find the average and variance of number in the system. State the conditions for the queue to be in equilibrium. Why is it not a good idea to operate a queuing system when arrival rate is close to service rate?

**(9+9)**

**3.**

a) A system (or a component) is observed at discrete points in time. Assume that the system is in operating states as running (state 0) or idle (state 2). If the system is undergoing repair (following breakdown), then the system state is denoted by state 1. A transition probability matrix of the system state is

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Draw the state diagram and compute $P^n$.

b) Let $Z_1$ and $Z_2$ be independent normally distributed random variables each having mean 0 and variance $\sigma^2$. Let $\lambda$ be a real constant and $X(t) = Z_1 \cos\lambda t + Z_2 \sin\lambda t$, $-\infty < t < \infty$. Find the mean and covariance functions of $X(t)$, $-\infty < t < \infty$ and show that it is a second order stationary process.

**(9+9)**

**4.**

a) Let *X* and *Y* be a random variables with sample space {1,2,3,4,5}.The joint distribution *p(x,y)* is given by

$$p(x,y) = \frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Obtain H(X), H(Y) and mutual information I(X:Y).

b) A coin having probability $p = 2/3$ of coming up heads is flipped 6 times. Compute the entropy of the outcome of this experiment.

**(12+6)**

**5.**

a) Evaluate: $L^{-1}\left(\dfrac{1}{\left(s+\sqrt{2}\right)\left(s-\sqrt{3}\right)}\right)$.

b) By the method of Laplace transform solve, the initial value problem:

$$y'' + 2y' + y = e^{-t},\ y(0) = -1 \text{ and } y'(0) = 1 .$$

c) Find the Fourier series of the function: $f(x) = x^2,\ -\pi < x < \pi.$

**(6+6+6)**

**6.**

a) Use inverse transform method to generate a random variable having probability density function:

$$f(x) = \frac{2x+1}{2}, 0 \le x \le 1$$

b) Show that relative error in Monte Carlo method is $O\left(n^{-1/2}\right)$.

c) A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$ where $x_i$ is equal to 1 if component *i* is working and is equal to 0 if component *i* is failed. Suppose that the system will work if components 1 and 2 are both working *or* if components 3 and 4 are both working. Let B be the event that the system will work. Specify all the outcomes in B and compute probability P(B).

**(6+6+6)**

**7.**

a) Solve the following Linear Programming problem using simplex method

Maximize $z = 7x_1 + 6x_2$

Subject to $2x_1 + x_2 \le 3$

$x_1 + 4x_2 \le 4$

$x_1, x_2 \ge 0$

b) Consider the following minimization problem:

Minimize $f(x) = x_1^2 + x_2^2 + x_3^2$

subject to

$g_1(X) = 2x_1 + x_2 - 5 \le 0$

$g_2(X) = x_1 + x_3 - 2 \le 0$

$g_3(X) = 1 - x_1 \le 0$

$g_4(X) = 2 - x_2 \le 0$

$g_5(X) = - x_3 \le 0$

obtain Kuhn-Tucker condition.

**(9+9)**